Question Sheet 7, Differentiation II. v1 2020-21

Mean Value Results

We look at applications of Rolle's Theorem, the Mean Value Theorem and Cauchy's Mean Value Theorem.

1. State Rolle's Theorem.

   Suppose that $f(x)$ is continuous on $[a,b]$, is twice differentiable on (a,b) and has three distinct zeros in $[a,b]$. Prove that there exists $c \in (a,b)$ such that $f''(c) = 0$.

   Can you generalise this result? Can you give a proof of your general result?

   (**Hint**: For a general result use proof by induction)

2. In a question on Sheet 4 it was seen that $e^x = 4x^2$ has at least 3 real solutions. Prove that it has **exactly** 3 real distinct solutions.

3. State the Mean Value Theorem.

   In each of the following examples find, if possible, an explicit value for $c \in (a,b)$ satisfying the *formula* of the Mean Value Theorem.

   In each case explain whether the Mean Value Theorem can be applied or not.

   i) $f(x) = x(x-2)(x-4), \quad \text{on } [1,3]$.

   ii) $f(x) = 1/x^2, \quad \text{on } [-1,1]$.

   iii) $f(x) = x^{1/3}, \quad \text{on } [-1,1]$.

4. i) Show that

$$\left|\sin^2 b - \sin^2 a\right| \leq |b - a|$$

for all $a, b \in \mathbb{R}$.

Careful. If you apply the Mean Value Theorem without thought you will get an unwanted factor of 2 on the right hand side.

ii) Show that

$$|\tan b - \tan a| \geq |b - a|$$

for all $a, b \in (-\pi/2, \pi/2)$.

iii) Show that

$$\cosh x \geq 1$$

for all $x \in \mathbb{R}$ and deduce

$$|\sinh b - \sinh a| \geq |b - a|$$

for all $a, b \in \mathbb{R}$.

iv) Show that

$$|\tanh b - \tanh a| \leq |b - a|$$

for all $a, b \in \mathbb{R}$.

5. i) Prove, by using the Mean Value Theorem, that

a) $\sin x$ is strictly increasing on $[-\pi/2, \pi/2]$,

b) $\cos x$ is strictly decreasing on $[0, \pi]$,

c) $\tan x$ is strictly increasing on $(-\pi/2, \pi/2)$.

ii) What result from the notes do you need to quote to justify the existence of a function

$$\arcsin : [-1, 1] \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right],$$

satisfying $\sin(\arcsin x) = x$ for all $x \in [-1, 1]$?

iii) Explain how to define

a) $\arccos : [-1, 1] \to [0, \pi]$,

b) $\arctan : \mathbb{R} \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

6. Using the Mean Value Theorem prove that

$$x - \frac{x^2}{2} < \ln(1+x) < x - \frac{x^2}{2} + \frac{x^3}{3}, \tag{1}$$

for $x > 0$.

(If necessary you may assume a result from the lectures that $e^x > 1 + x$ for all $x \neq 0$.)

Hint Define an appropriate functions of t on the interval $[0, x]$, one for the lower bound in (2) the other for the upper.

7. State the Cauchy Mean Value Theorem.

Assume that f is continuous on $[a, b]$ and differentiable on (a, b). Prove that

i) there exists $c \in (a, b)$ such that

$$f(b) = f(a) + \left(e^{b-c} - e^{a-c}\right) f'(c).$$

Hint Rearrange this equality so that a and b occur on one side, c on the other. You should then deduce what function has to be chosen for g in Cauchy's Mean Value.

ii) Assuming further that $f(x) > 0$ and $f'(x) \neq 0$ on $[a, b]$ prove that there exists $c \in (a, b)$ such that

$$f(b) = f(a) + \ln\left(\frac{f(b)}{f(a)}\right) f(c).$$

Strangely no derivatives appear anywhere!

iii) Assuming that $f'(x) \neq 0$ for all $x \in (a, b)$ prove that there exists $c \in (a, b)$ such that

$$f(b) = f(a) + e^{f(b)-f(c)} - e^{f(a)-f(c)}.$$

Again no derivatives appear.

Inverses

We have proven the existence of the inverse of trig functions in Question 5 and of the hyperbolic trig functions in Question 7, Sheet 5. We now ask if they are differentiable.

8. State the Theorem on the Differentiation of Inverse functions.

   i) Prove that
   $$\cos(\arcsin y) = \sqrt{1-y^2}$$
   for $-1 < y < 1$.

   Hint: Try to use $y = \sin(\arcsin y)$ by relating cos to sin through $\sin^2\theta + \cos^2\theta = 1$ for all θ.

   ii) Use the Theorem on the Differentiation of Inverse functions to prove that
   $$\frac{d}{dy}\arcsin y = \frac{1}{\sqrt{1-y^2}}$$
   on $(-1, 1)$,

   iii) Give similar results for

   a) $\dfrac{d}{dy}\arccos y \quad \text{on } (-1,1)$,

   b) $\dfrac{d}{dy}\arctan y \quad \text{on } \mathbb{R}$.

9. Prove that
   $$\arctan x > \frac{x}{1 + x^2/3}$$
   for $x > 0$.

   Hint This question is here as an application of the derivative of arctan found in the previous question.

10. i) a) Prove that
   $$\cosh\left(\sinh^{-1} y\right) = \sqrt{1+y^2}$$
   for all $y \in \mathbb{R}$.

b) Use the Theorem on the Differentiation of Inverse functions to prove that

$$\frac{d}{dy}\sinh^{-1} y = \frac{1}{\sqrt{1+y^2}}$$

for all $y \in \mathbb{R}$.

ii) Prove that

$$\frac{d}{dy}\cosh^{-1} y = \frac{1}{\sqrt{y^2-1}}$$

for $y > 1$.

iii) Evaluate

$$\frac{d}{dy}\tanh^{-1} y$$

for $y \in (-1, 1)$.

L'Hôpital's and Chain Rule

11. State L'Hôpital's Rule.

Use L'Hôpital's Rule to evaluate

i)

$$\lim_{x\to 0} \frac{\ln(1-x) + \ln(1+x)}{x^2},$$

ii)

$$\lim_{x\to 0} \frac{(1+x)\ln(1-x) - (1-x)\ln(1+x) + 2x}{x^3},$$

iii)

$$\lim_{x\to 0} \frac{\arcsin x - x}{x^3},$$

iv)

$$\lim_{x\to 0} \frac{\sinh^{-1} x - x}{x^3}.$$

12. State the Chain Rule for Differentiation.

Prove

i)

$$\frac{d}{dy}\left(\tanh^{-1}(\sin y)\right) = \frac{1}{\cos y}$$

for $y \in (-\pi/2, \pi/2)$.

ii)

$$\frac{d}{dy}\left(\sinh^{-1}(\tan y)\right) = \frac{1}{\cos y}$$

for $y \in (-\pi/2, \pi/2)$.

iii) Can you give a similar example for $\cosh^{-1}$ with an appropriate trigonometric function?

Additional Questions

13. Using the Mean Value Theorem prove that

$$\frac{1}{1 - x + \dfrac{x^2}{3}} > e^x > \frac{1}{1 - x + \dfrac{x^2}{2}},$$

the first inequality for $0 < x < 1$, the second for all $x > 0$.

Hint Examine each inequality separately and multiply up.

14. An example of the use of L'Hôpital's Rule, stated but left to the student to check, was

$$\lim_{x \to 0} \frac{\sin x - x}{x^3} = -\frac{1}{6}.$$

Use this result to show that the function

$$f(x) = \begin{cases} \dfrac{\sin x}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0, \end{cases}$$

is differentiable at $x = 0$ and find the value of the derivative at $x = 0$.